A reply to Scutaru's letter on the generalised exponents of sl(3,C)



1990 J. Phys. A: Math. Gen. 23 1827

(http://iopscience.iop.org/0305-4470/23/10/022)

J. Phys. A: Math. Gen. 23 (1990) 1827-1828. Printed in the UK

COMMENT

# A reply to Scutaru's letter on the generalised exponents of sl(3, $\mathbb{C}$)

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Received 26 October 1989, in final form 9 January 1990

**Abstract.** In a recent letter on the generalised exponents of the Lie algebra sl(3, $\mathbb{C}$), Scutaru refers to an integrity basis of the enveloping algebra proposed by us as a hypothesis and then proceeds to establish its validity. The purpose of this comment is to stress the fact that this integrity basis was not a hypothesis but a proven result.

Using a result of Kostant [1], Scutaru [2] recently derived a formula for the generalised exponents of the Lie algebra sl(3, $\mathbb{C}$). He then proceeded to establish a one-to-one correspondence between the sl(3, $\mathbb{C}$)-module structure of the enveloping algebra of sl(3, $\mathbb{C}$) as described by his formula and the one implied by the integrity basis proposed by us [3]; he refers to this basis as a hypothesis and concludes that the above exercise proves its validity. The purpose of this comment is to point out that the generating function derived in [3]

$$\mathscr{G}(U;\Lambda_1,\Lambda_2)=\frac{1+U^2\Lambda_1\Lambda_2+U^4\Lambda_1^2\Lambda_2^2}{(1-U^2)(1-U^3)(1-U\Lambda_1\Lambda_2)(1-U^3\Lambda_1^3)(1-U^3\Lambda_2^3)} \tag{1}$$

where $U$ carries the degree of the tensor operator and $\Lambda_i$ the representation labels $\lambda_i$ (Cartan labels) as exponents, already establishes this result and therefore, as far as this one-to-one correspondence is concerned, the integrity basis in [3] was a proven result rather than a hypothesis.

Given a Lie algebra $L$ of a Lie group G, its enveloping algebra V($L$) may be expressed [4] as the direct sum

$$\mathrm{V}(L)=\mathrm{V}^0(L)+\mathrm{V}^1(L)+\mathrm{V}^2(L)+\ldots+\mathrm{V}^n(L)+\ldots \tag{2}$$

where $\mathrm{V}^n(L)$ represents the set of elements of V($L$) which are symmetric and homogeneous of degree $n$, and $\mathrm{V}^0(L)\equiv\mathbb{C}$. Let S($L$) be the symmetric algebra of $L$ Since V($L$) and S($L$) are isomorphic as $L$-modules, the decomposition by degree of V($L$) into a direct sum of irreducible $L$-modules which respects the grading (2) is identical to the decomposition of the tensor products

$$(\Gamma)^n=\Gamma\otimes\Gamma\otimes\ldots\otimes\Gamma \qquad n \text{ identical copies } (n=1,\infty) \tag{3}$$

where $\Gamma$ is a $d$-dimensional tensor whose components are $c$-numbers and which transforms by the adjoint representation of $L$. Kostant's generalised exponents correspond to the degrees of the irreducible tensors into which the products (3) decompose.

The decomposition (3) is known as the symmetric plethysm and the irreducible tensors are referred to as polynomial tensors based on $\Gamma$; the corresponding tensor operator in $V(L)$ is obtained through symmetrisation with respect to order. We shall now briefly describe the method used in [3] to decompose (3).

This method makes use of a subgroup H of G. The tensor $\Gamma$ is a reducible tensor of H. Let us denote by $\Gamma^1_H, \Gamma^2_H, \ldots, \Gamma^m_H$ the $m$ irreducible H-tensors into which $\Gamma$ decomposes. The idea is to first construct a generating function for $H$-tensors based on the $m$ tensors $\Gamma^i_H$ and then transform it into a generating function for G-polynomial tensors based on $\Gamma$. If one chooses $H \equiv U(1) \times U(1) \ldots \times U(1)$ $d$ times, the generating function for H-tensors is simply the generating function for weights; the transformation is then effected by making use of Weyl's characteristic function. The generating function (1) is derived by using $H \equiv SU(2) \times U(1)$; the conversion from a generating function for H-tensors to a generating function for G-tensors was done using the group–subgroup characteristic function whose role in the conversion parallels that of the Weyl characteristic function in converting a weight generating function to a generating function for G-tensors. In this approach one first derives the generating function for polynomial tensors ((1) in the $sl(3, \mathbb{C})$ case) without any assumption about the integrity basis; at each step of the derivation, generating functions with proven integrity bases are used and since everything is done analytically, the generating function (1) constitutes a proven result and the validity, as far as the one-to-one correspondence which Scutaru establishes in [2] is concerned, of the integrity basis and syzygies follows immediately. There is no need for further testing, although the work of Scutaru does establish in a slightly more explicit form the syzygy written in symbolic form in (3.7) of our earlier paper.

## References

[1] Kostant B 1963 *Am. J. Math.* **85** 327
[2] Scutaru H 1989 *J. Phys. A: Math. Gen.* **22** L191
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